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Application of Euler–Maclaurin sum formula to obtain an approximate closed-form Green's function for a two-dimensional acoustical space

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Abstract

The Green's function solution for the acoustic wave equation in a two-dimensional rectangular space is expressed as an infinite series of terms based on the cross modes of the duct. An approximate closed-form solution is obtained by applying the Euler–Maclaurin sum formula. The procedure provides a closed-form expression in both the space–time and Laplace domains along with an upper bound for a remainder. Plane wave and higher-order waves components are identified. A numerical example for an exponential input gives comparisons of the transient response for the approximate closed form and series Green's function solutions. The time response and analytical transfer function frequency spectrum of the series and Euler–Maclaurin closed-form Green's function are computed. Lastly, the approximate closed-form Green's transfer function expression is used in model-based designed feedforward and feedback control schemes to reduce peaks in the frequency response and provide system damping.

1. Introduction

The solution for the Green's function for the acoustic wave equation in a two-dimensional (2D) space in terms of an infinite series of cross mode terms has been given by Fahy [1] and Morse and Ingard [2]. In Ref. [1], the solution is given as a harmonic response in the space–frequency domain while in Ref. [2], the solution is given as a space–time response. The characteristics of the individual modal terms are clearly defined in each case but the result is still an infinite series. Zimmer et al. [3] provided an infinite series for the acoustic pressure in three-dimensional (3D) wave-guides with two finite boundaries and one semi-infinite boundary (i.e. an acoustical tunnel or duct). They performed extensive numerical calculations to obtain a sum for a truncated series for a practical application. The use of the Euler–Maclaurin sum formula for obtaining an approximate numerical value for the sum of an infinite number of terms has received a great deal of attention in the past. A few appropriate examples are discussed here. The application of the Euler–Maclaurin sum formula to obtain a numerical result for infinite series has been presented by Riesel [4] where an approximate solution to a

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double infinite sum has been calculated. The application of the Euler–Maclaurin sum for summing numerical infinite series in quantum physics problems has also had some activity. Kao et al. [5] computed the total number of particles interacting by summing an expression in a statistical thermodynamics problem. Fernandez and Pineiro [6] computed the sum of quantum numbers in problem where large number of energy levels are included. As with mostly all applications of the Euler–Maclaurin sum formula, the references cited above dealt with numerical problems resulting in a specific numerical value.

The direct application of the Euler–Maclaurin sum formula to a series defined in space–time has received little attention in the past. The summation of a space–time series is the underlying concept of its use for this paper and for several other papers by the author. Panza [7] obtained a closed-form space–time solution for the Green's function in a partially bounded acoustical space by applying the Euler–Maclaurin sum formula to the infinite modal series in the space-Laplace transform domain describing the system. He also applied the result to determine the space–time response to an exponential time function input with a method of extension to an arbitrary time function. Additionally, Panza [8] applied the Euler–Maclaurin sum formula to derive a closed-form expression for the infinite series representation of Green's function that occurs for a finite length cable beam or string. He then derived finite controllers in the Laplace transform domain for both an acoustic space and a cable beam and presented frequency-domain results. Recently, Panza [9] applied the Euler–Maclaurin sum formula to a harmonic series of modes in the 2D duct to predict the attenuation of a dissipative silencer.

In this paper, an approximate closed-form expression for Green's function is obtained by applying the Euler–Maclaurin sum formula directly to the infinite series form of Green's function in the space–time domain. The method is different from that in Refs. [7,8] where the direct application is in the space-Laplace transform domain and in Ref. [9] where the series results from a harmonic excitation. Additionally, the 2D region for this paper gives an extra space bound over the regions in Refs. [7,8] resulting in more complex functions arising from the summing procedure. The approximate closed-form Green's function is compared to the infinite series form in both the time and frequency domains. The approximate closed-form Green's function is applied to determine the transient response to an exponential input and the frequency response for a model-based feedforward and a feedback active control scheme.

2. Green's function modal expansion

Consider a 2D acoustic space x, z bounded by perfectly reflective surfaces at $z = 0$ and H. Practical examples of such a region include a thin rectangular duct and an acoustical tunnel. Green's function $g(x,z,t)$ is the solution to the partial differential wave equation

$$\frac{\partial^2 g(x,z,t)}{\partial x^2} + \frac{\partial^2 g(x,z,t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 g(x,z,t)}{\partial t^2} = -\delta(x-x_o)\delta(z-z_o)\delta(t), \tag{1}$$

where c is the speed of sound and boundary conditions are given as

$$\frac{\partial g(x,z,t)}{\partial z} = 0 \text{ at } z = 0 \text{ and } H. \tag{2}$$

The modal expansion that satisfies the partial differential equation (PDE) and boundary conditions is given by

$$g(x,z,t) = \sum_{n=0}^{\infty} \cos\left(\frac{n\pi z}{H}\right) q_n(x,t). \tag{3}$$

Defining $G(x,z,s)$ as the Laplace transform from time t to s, multiplying by $\cos(m\pi z/H)$, integrating over z, and using modal orthogonality, an infinite number of modal ordinary differential equations for the Laplace transform $Q_n(x,s)$ are given by

$$\frac{\mathrm{d}^2 Q_n(x,s)}{\mathrm{d}x^2} - \left[\left(\frac{n\pi}{H}\right)^2 + \left(\frac{s}{c}\right)^2\right] Q_n(x,s) = -\frac{2}{H}\delta(x-x_o)\cos\left(\frac{n\pi z_0}{H}\right), \quad n = 0,1,2,\ldots,\infty. \tag{4}$$

Defining $Q_n(\zeta,s)$ as the Fourier transform from space x to ζ and applying the Fourier transform to Eq. (4) gives

$$Q_n(\varsigma, s) = \frac{2}{H} \frac{e^{-jx_o\varsigma} \cos\left(n\pi z_o/H\right)}{\varsigma^2 + \left[\left(n\pi/H\right)^2 + \left(s/c\right)^2\right]}. \tag{5}$$

Applying the inverse Fourier transform from ζ to x gives

$$Q_n(x, s) = \frac{1}{H} \frac{e^{-\sqrt{\left(n\pi/H\right)^2 + \left(s/c\right)^2}|x - x_o|} \cos\left(n\pi z_0/H\right)}{\sqrt{\left(n\pi/H\right)^2 + \left(s/c\right)^2}}. \tag{6}$$

In previous work [7,8], the application of the Euler–Maclaurin sum formula to a modal sum in the Laplace domain s, as represented by

$$G(x, z, s) = \sum_{n=0}^{\infty} \cos\left(\frac{n\pi z}{H}\right) Q_n(x, s), \tag{7}$$

resulted in an integral with an exact solution. However, this is not the case for the specific $Q_n(x,s)$ given in Eq. (6). For this paper, the modal expansion will first be transformed from the Laplace domain s to the time domain. Inverting Eq. (6) gives Green's function as an infinite series of Bessel functions:

$$g(x, z, t) = \frac{c}{H} \sum_{n=0}^{\infty} J_o\left(\frac{n\pi c}{H} \sqrt{t^2 - \left(\frac{x - x_o}{c}\right)^2}\right) U\left(t - \frac{|x - x_o|}{c}\right) \cos\left(\frac{n\pi z_o}{H}\right) \cos\left(\frac{n\pi z}{H}\right), \tag{8}$$

where U is the unit Heaviside step function.

Letting

$$\tau = \sqrt{t^2 - \left(\frac{x - x_0}{c}\right)^2}$$

and expanding the cosine product gives a relative Green 's function

$$g_R(x, z, t) = \frac{2Hg(x, z, t)}{cU\left(t - \frac{|x - x_o|}{c}\right)} = \sum_{n=0}^{\infty} J_0\left(\frac{n\pi c\tau}{H}\right) \cos\left[\frac{n\pi(z - z_0)}{H}\right] + \sum_{n=0}^{\infty} J_0\left(\frac{n\pi c\tau}{H}\right) \cos\left[\frac{n\pi(z + z_0)}{H}\right]. \tag{9}$$

3. Application of Euler–Maclaurin sum formula

The basic form of the formula is given by Apostal [10]

$$\sum_{n=0}^{\infty} f_n = \int_0^{\infty} f(\mu)\, d\mu + \left(\frac{1}{2}\right)[f(0) + f(\infty)] - \int_0^{\infty} \frac{\partial f}{\partial \mu} \sum_{k=1}^{\infty} \frac{\sin(2\pi k\mu)\, d\mu}{\pi k}\, d\mu. \tag{10}$$

Defining $z_1 = z + z_o$, $z_2 = z - z_o$ and applying Eq. (10) to the sums in Eq. (9), with the order of integration and summation switched, gives

$$g_R(x, z, t) = \frac{2Hg(x, z, t)}{cU\left(t - \frac{|x - x_o|}{c}\right)} = \sum_{i=1}^{2} \int_0^{\infty} J_o\left(\frac{\mu\pi c\tau}{H}\right) \cos\left(\frac{\mu\pi z_i}{H}\right) d\mu + 1 + R(x, z, t), \tag{11}$$

where R is given by

$$R(x,z,t) = -\sum_{i=1}^{2}\sum_{k=1}^{\infty}\frac{1}{\pi k}\int_0^\infty \frac{\partial}{\partial\mu}\left[J_o\left(\frac{\mu\pi c\tau}{H}\right)\cos\left(\frac{\mu\pi c z_i}{H}\right)\right]\sin(2\pi k\mu)\,\mathrm{d}\mu. \tag{12}$$

Applying integration by parts to the integral in Eq. (12), all of the integrals in Eqs. (11) and (12) have solutions from Bowman [11] given as

$$\int_0^\infty J_o(\mu a)\cos(\mu b)\,\mathrm{d}\mu = \frac{U(a-b)}{\sqrt{a^2-b^2}} + \frac{U(b-a)}{\sqrt{b^2-a^2}}, \tag{13}$$

where the Heaviside step function U allows for all possibilities of a and b.

Eq. (11) can be written as

$$g_R(x,z,t) = 1 + \sum_{i=1}^{2}\left[\frac{U\left(\frac{\pi c\tau}{H}-\frac{\pi z_i}{H}\right)}{\sqrt{\left(\frac{\pi c\tau}{H}\right)^2-\left(\frac{\pi z_i}{H}\right)^2}} + \frac{U\left(\frac{\pi z_i}{H}-\frac{\pi c\tau}{H}\right)}{\sqrt{\left(\frac{\pi z_i}{H}\right)^2-\left(\frac{\pi c\tau}{H}\right)^2}}\right] + R(x,z,t), \tag{14}$$

with $R(x,z,t)$ from Eq. (12) written as

$$\begin{aligned} R(x,z,t) = {} & \sum_{i=1}^{2}\sum_{k=1}^{\infty}\left[\frac{U\left(\frac{\pi c\tau}{H}-\frac{\pi z_i}{H}+2\pi k\right)}{\sqrt{\left(\frac{\pi c\tau}{H}\right)^2-\left(\frac{\pi z_i}{H}-2\pi k\right)^2}} + \frac{U\left(\frac{\pi z_i}{H}-2\pi k-\frac{\pi c\tau}{H}\right)}{\sqrt{\left(\frac{\pi z_i}{H}-2\pi k\right)^2-\left(\frac{\pi c\tau}{H}\right)^2}}\right] \\ & + \sum_{i=1}^{2}\sum_{k=1}^{\infty}\left[\frac{U\left(\frac{\pi c\tau}{H}-\frac{\pi z_i}{H}-2\pi k\right)}{\sqrt{\left(\frac{\pi c\tau}{H}\right)^2-\left(\frac{\pi z_i}{H}+2\pi k\right)^2}} + \frac{U\left(\frac{\pi z_i}{H}+2\pi k-\frac{\pi c\tau}{H}\right)}{\sqrt{\left(\frac{\pi z_i}{H}+2\pi k\right)^2-\left(\frac{\pi c\tau}{H}\right)^2}}\right]. \end{aligned} \tag{15}$$

Eqs. (14) and (15) provide an exact solution for Green's function that is a different form than the modal representation of Eq. (8). It has the appearance of an images sum form but is not the actual sum of the infinite number of line source images between the planes perpendicular to the z direction. The constant term is one-half of the plane wave component and the maximum value of the remainder infinite sum is shown below to be the other half of the plane wave component. The two finite terms represent the majority of the higher-order modal reverberation or in essence the primary contribution of the many images due to the two parallel reflecting planes. Although image like in form, they are not just the first two source images due to the parallel planes. The remainder terms with the summation over index k are essentially higher-order versions of the two finite terms. A similar result is given by Panza [8] for the case of two parallel plates.

The finite number of terms in Eq. (14) (i.e. other than the $R(x,z,t)$ series) comprises the main part of the Euler–Maclaurin sum. The infinite sums in $R(x,z,t)$ of Eq. (15) are useful for showing the connection between their specific form and the Bessel function form of the modal terms and for showing the basic role of the Euler–Maclaurin sum formula in providing this connection. For practical computational and design uses intended for this paper, a finite number of terms are required with an approximation for the remainder $R(x,z,t)$. Apostal [10] gives another more useful form for the Euler–Maclaurin sum formula given by

$$\sum_{n=0}^{\infty} f_n = \int_0^\infty f(\mu)\,\mathrm{d}\mu + \left(\frac{1}{2}\right)[f(0)+f(\infty)] + \int_0^\infty \frac{\partial f}{\partial\mu}\left(\mu-[\mu]-\frac{1}{2}\right)\mathrm{d}\mu, \tag{16}$$

where [ ] represents the smallest integer part. The second integral is the term $R(x,z,t)/2$ because there are two sums in the modal expansion of Eq. (9).

The term in the second integral in Eq. (10) is a staircase function which represents the infinite series. The goal of this paper is obtain an approximate closed-form solution that has only a few terms so that it may be used as a model in applications instead of a truncated version of the infinite series of Eq. (9) where many terms may be required. Panza [7] has shown that the second integral term in Eq. (16) can be viewed as remainder with an upper bound. The upper bound of $R(x,z,t)$ is a remainder given by

$$R_{\text{Max}} = 2\left[\begin{array}{c}\left|\int_0^\infty \frac{\partial f}{\partial \mu}\left(\mu - [\mu] - \frac{1}{2}\right) \mathrm{d}\mu\right| \leqslant \text{Max}\left|\mu - [\mu] - \frac{1}{2}\right|\left|\int_0^\infty \frac{\partial f}{\partial \mu}\mathrm{d}\mu\right| \\ = \frac{1}{2}\left|\int_0^\infty \mathrm{d}f\right| = \frac{1}{2}\left|f(\infty) - f(0)\right| = \frac{1}{2}\end{array}\right] = 1. \tag{17}$$

The approximate relative Green's function in Eq. (14) is given by

$$g_R(x,z,t) = 1 + \sum_{i=1}^{2}\left[\frac{U\left(\frac{\pi c\tau}{H} - \frac{\pi z_i}{H}\right)}{\sqrt{\left(\frac{\pi c\tau}{H}\right)^2 - \left(\frac{\pi z_i}{H}\right)^2}} + \frac{U\left(\frac{\pi z_i}{H} - \frac{\pi c\tau}{H}\right)}{\sqrt{\left(\frac{\pi z_i}{H}\right)^2 - \left(\frac{\pi c\tau}{H}\right)^2}}\right] + R_{\text{Max}}. \tag{18}$$

From the series in Eq. (9), the $n = 0$ terms in the Bessel functions represent the plane wave component given as $g_R = 2$. Thus the Euler–Maclaurin sum of Eq. (18) provides the plane wave as $g_R = 1 + R_{\text{Max}} = 2$ with the two term sum giving an approximate closed-form representation of all of the higher-order modes.

4. Applications with numerical examples

4.1. Transient response

The Euler–Maclaurin solution is expected to apply to the general case of system dimensions. A typical example is presented here. The transient behavior of the Green's function is given for a numerical case where $H = 3.1$ m, $x - x_o = 9.1$ m, and $z = z_o = 0$. The 3.1 m height may represent an acoustical tunnel or large duct where higher-order modes start to become significant. The relative Green's function times the Heaviside step $U\left(t - \left(|x - x_o|/c\right)\right)$ is computed. This gives the result $(2H/c)g(x,z,t)$. Twenty terms are sufficient for convergence in the series sum of Eq. (9) to show the contrast between the series and Euler–Maclaurin sum solutions. Fig. 1 gives a comparison of the series solution of Eq. (9) with the approximate closed-form Euler–Maclaurin sum solution of Eq. (18). For all Euler–Maclaurin sum results given in this paper, remainder term $R_{\text{Max}} = 1$ is included because it provides one-half of the plane wave. For a full time range in Fig. 1(a), the Euler–Maclaurin solution gives a smooth monotonically decreasing response and reaches a steady-state value of two as obtained from the series solution. The value of two represents the plane wave component. The peak value at $t-(x/c)$ is very large and is not shown on the plot. The series solution gives a series of transient peaks all along the time span while approaching the plane value of two. Fig. 1(b) gives a closer look at the higher-order terms (i.e. excluding the plane wave) for the early part of the process in the vicinity of $t = x/c$. Here it is seen that the higher-order part of the Euler–Maclaurin solution does reasonably match that for the series solution. The advantage of the Euler–Maclaurin solution is that it gives only a few terms compared to much more required for the series solution. In fact, for the specific case of $z = z_o = 0$, only two terms in Eq. (18)

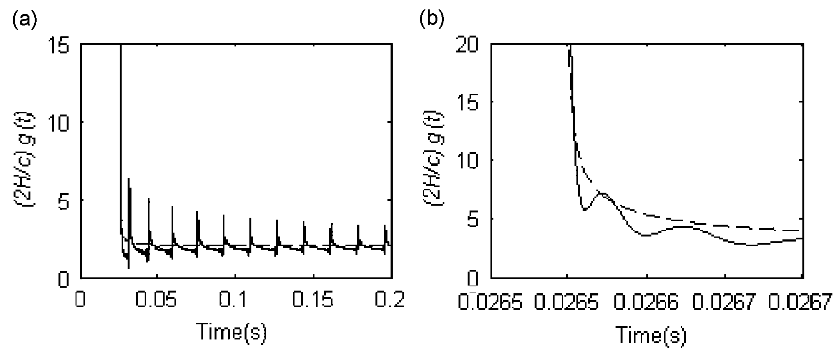

Fig. 1. Green's function Euler–Maclaurin sum closed-form solution (- - - -) vs. series solution (——): (a) full time response and (b) early part of the transient.

form the approximate Green's function compared to 21 modal terms required for the series form in Eq. (9). The approximate solution provides a trend similar to the series solution but does not contain the small oscillatory peaks occurring after the initial peak near $t = x/c$. However, the approximate solution matches the series solution very well in the vicinity of $t = x/c$ where the Green function values are very large. The primary purpose of the approximate solution is to use it for the predicting transient response to a pulse type input and to use it in model-based controllers. For the former, since determining the response to a particular transient input requires the Green's function to be used in a convolution with the input, it is shown in this paper that the Euler–Maclaurin approximate closed-form expression matches the infinite series close enough for it to be a reasonable alternative. The convolution process with a bounded input tends to smooth out ripples in the Green's function.

A time decaying exponential input is first used to demonstrate the accuracy of the Euler–Maclaurin solution. Convolution with a volume velocity input proportional to an exponential $f(t) = \mathrm{e}^{-\alpha t}$ requires the derivative of the Green's function and is given by

$$y(t) = \int_0^t f(\tau)\frac{\mathrm{d}g(t-\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau. \tag{19}$$

Integration by parts for the exponential input gives

$$y(t) = -\mathrm{e}^{-\alpha t}g(0) + g(t) - \alpha\int_0^t \mathrm{e}^{-\alpha\tau}g(t-\tau)\,\mathrm{d}\tau. \tag{20}$$

For g, we will use the form $g_R(x,z,t)U\big(t - \big(|x - x_o|/c\big)\big)$ as in the transient Green's function results above. Eq. (9) is used for the series solution and Eq. (18) is used for the Euler–Maclaurin solution. The Euler–Maclaurin solution can be expected to give a good representation of the transient decay over time because the convolution integral tends to smooth out the peaks and valleys that are present in the series solution. For the case where $z = z_o = 0$, the integral in Eq. (20) is given by

For the infinite series,

$$\int_0^t \mathrm{e}^{-\alpha\tau}g(t-\tau)\,\mathrm{d}\tau = 2\sum_{n=0}^{\infty}\int_0^{t-(x/c)} \mathrm{e}^{-\alpha\tau}J_o\left(\frac{n\pi c}{H}\sqrt{(t-\tau)^2 - \left(\frac{x}{c}\right)^2}\right)\mathrm{d}\tau\, U\left(t - \frac{x}{c}\right). \tag{21}$$

For Euler–Maclauirn closed-form sum with remainder equal to one,

$$\int_0^t \mathrm{e}^{-\alpha\tau}g(t-\tau)\,\mathrm{d}\tau = \left\{\frac{2}{\alpha}\left[1 - \mathrm{e}^{-\alpha(t-(x/c))}\right] + 2\int_0^{t-(x/c)}\frac{\mathrm{e}^{-\alpha\tau}}{(\pi c/H)\sqrt{(t-\tau)^2 - (x/c)^2}}\,\mathrm{d}\tau\right\}U\left(t - \frac{x}{c}\right). \tag{22}$$

Fig. 2(a) shows that for the case of $H = 3.1$ m, $x - x_o = 9.1$ m, and $z = z_o = 0$, the agreement between the series and the closed-form Euler–Maclaurin sum is very good for $t \geqslant 0.1$ s. Fig. 2(b) shows that after a short

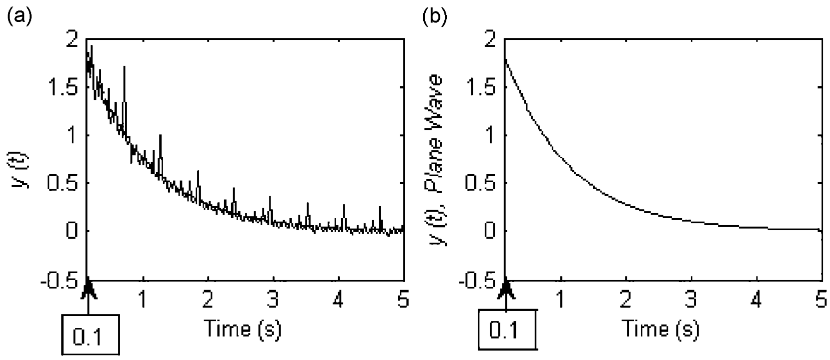

Fig. 2. Transient response to exponential input for $t \geqslant 0.1$ s for Euler–Maclaurin sum closed-form solution (- - - -) vs. series solution (——): (a) all terms included and (b) plane wave term only.

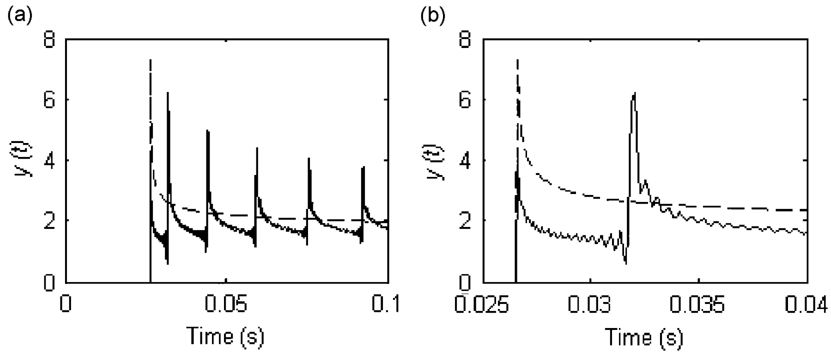

Fig. 3. Early part of higher-order terms transient response to exponential input for Euler–Maclaurin sum closed-form solution (- - - -) vs. series solution (——): (a) for $t \leqslant 0.1$ s and (b) for $t \leqslant 0.04$ s.

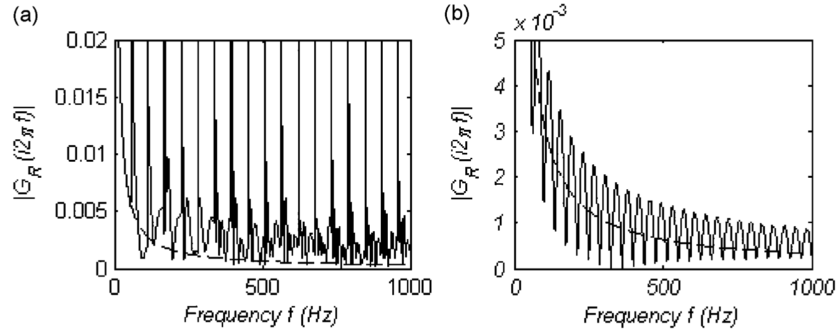

Fig. 4. Magnitude of frequency response function generated from the Green's function (——) and from the plane wave transfer function (- - - -): (a) series solution and (b) Euler–Maclaurin sum closed-form solution.

time, the plane wave contribution dominates the response. Thus the small peaks in Fig. 1(a) after $t = 0.1$ s due to the series terms may be considered insignificant. Fig. 3(a) gives a comparison of the higher-order components for the time range from $t = 0$ to 0.1 s just after the input is applied The response using the one term in the Euler–Maclaurin approximate Green's function provides the same basic trend as the 20 higher-order mode terms (i.e. total less plane wave) in the series response. Additionally, Fig. 3(b) shows that the peak value of 7.3 for the Euler–Maclaurin solution is close to the peak value of 6.2 for the series solution. The time shift occurrence of peak values of approximately 0.005 s may be considered insignificant.

4.2. Frequency response

The frequency response is obtained from the Laplace transform of Green's function. For the series solution, $s = \mathrm{i}\omega$ is substituted into Eqs. (6) and (7). For the Euler–Maclaurin solution, the case of $z = z_0 = 0$ gives a relative Green's function expressed as

$$g_R(t) = 1 + R_{\text{Max}} + 2\frac{U\left[t - \left(x - x_o/c\right)\right]}{\left(\pi c/H\right)\sqrt{t^2 - \left(x - x_o/c\right)^2}}. \tag{23}$$

From Chu et al. [12], the Laplace transform is given by

$$G_R(s) = [1 + R_{\text{Max}}]\frac{1}{s} + \frac{2H}{\pi c}K_o\left[\left(\frac{x - x_o}{c}\right)s\right], \tag{24}$$

where K_o is the modified Bessel function of the second kind.

Fig. 4(a) gives the magnitude of the series system frequency response function from Eqs. (6) and (7) and Fig. 4(b) gives the magnitude of the Euler–Maclaurin system frequency response function from Eq. (24) with $s = i\omega$. The monotonically decreasing plane wave transfer function is included for reference. The magnitude of the series system frequency response function contains uniformly spaced large resonance peaks (with actual values much greater than shown) at the cross mode resonance frequencies $f_n = nc/2H$ and a level of random like peaks in the vicinity of the plane wave. The magnitude of the Euler–Maclaurin system frequency response function contains uniformly space bounded peaks somewhat symmetrical to the plane wave. The spacing of the peaks is approximately 50% closer than cross mode resonance frequencies. The fact that one term of the Euler–Maclaurin sum provides a frequency response function with higher-order mode oscillatory behavior somewhat similar in form to the to the 20-term series higher-order mode sum indicates that it may be a more useful and much simpler substitute for designing control systems for providing damping to reduce the large resonance peaks inherent in an acoustic space that initially has very little dissipation. To show the utility of the simple Euler–Maclaurin transfer function, two control schemes are presented. The model-based nature of the schemes means that the Euler–Maclaurin transfer function model is used in the controller design while the actual transfer function from the infinite series is used as the actual system simulation.

One scheme is based on a simple model-based feedforward control scheme designed with the Euler–Maclaurin transfer function to provide a desired output performance over a broad range of frequencies when an input disturbance $F_d(s)$ excites the system. Fig. 5 shows a block diagram schematic for the system where a measurement of the disturbance is required. Although the disturbance measurement in this scheme may be difficult to practically implement, it is presented here to demonstrate another way to utilize the finite Euler–Maclaurin sum. We define $G_S(s)$ as the actual plant transfer function based on the infinite series form of $g_R(t)$ (i.e. the Laplace transform in Eqs. (6) and (7)) and $G_{\mathrm{EM}}(s)$ as the closed-form plant transfer function based on the Euler–Maclaurin form for $g_R(t)$ (i.e. from the Laplace transform in Eq. (24)). We also define a control input $U(s)$ proportional to the disturbance giving $U(s) = G_{\mathrm{FF}}(s)F_d(s)$ where $G_{\mathrm{FF}}(s)$ is the feedforward controller transfer function. The system output $Y(s)$ for feedforward control is given in terms of the series transfer function by

$$Y(s) = U(s)G_S(s) + F_d(s)G_S(s). \tag{25}$$

For the controller design, consider a desired output given in terms of the Euler–Maclaurin closed-form transfer function given by

$$Y_{\mathrm{des}}(s) = U(s)G_{\mathrm{EM}}(s) + F_d(s)G_{\mathrm{EM}}(s) = \mathrm{TF}_{\mathrm{des}}(s)F_d(s), \tag{26}$$

where $\mathrm{TF}_{\mathrm{des}}(s)$ is a desired transfer function for the system.

Solving Eq. (26) for $U(s)$ and substituting into $U(s) = G_{\mathrm{FF}}(s)F_d(s)$ gives a finite form for the feedforward controller transfer function in terms of the closed-form model G_{EM}(s). This form is given as

$$G_{\mathrm{FF}}(s) = \frac{\mathrm{TF}_{\mathrm{des}}(s)}{G_{\mathrm{EM}}(s)} - 1. \tag{27}$$

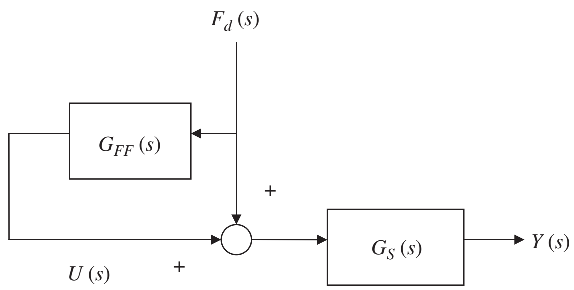

Fig. 5. Block diagram schematic for feedforward control scheme.

Using $G_{FF}(s)$ from Eq. (27) in $U(s) = G_{FF}(s)F_d(s)$ and substituting into Eq. (25), the actual system output transfer function with control is given as

$$\mathrm{TF}_{\mathrm{actual}}(s) = \frac{Y(s)}{F_d(s)} = \mathrm{TF}_{\mathrm{des}}(s)\frac{G_S(s)}{G_{\mathrm{EM}}(s)}. \tag{28}$$

Thus if one uses the actual plant transfer function as the control model plant transfer function (i.e. $G_{EM}(s) = G_S(s)$), the actual system output transfer function would equal the desired transfer function (i.e. $\mathrm{TF}_{\mathrm{actual}}(s) = \mathrm{TF}_{\mathrm{des}}(s)$). However, the feedforward control transfer function $G_{FF}(s)$ would be in principle an infinite series of terms and would not be practical to implement even in a truncated form of at least 21 terms that would be required for convergence. The advantage of using a finite Euler–Maclaurin sum result (with only one term for higher-order modes) for the plant transfer function in the model-based controller is the use of a finite feedforward controller transfer function. Fig. 6(a) gives a comparison of the magnitude of the system frequency response function with control based on the finite Euler–Maclaurin sum and the desired transfer function. The desired frequency response of the system is set to be constant $|\mathrm{TF}_{\mathrm{des}}(s)| = 0.0001$ which drives the controlled frequency response magnitude to be in this neighborhood. The vertical scale of Fig. 6(a) is the same as the series transfer function in Fig. 4(a) to allow a direct comparison of controlled and uncontrolled systems. This comparison indicates a significant reduction of magnitudes obtained with the finite Euler–Maclaurin-based feedback controller. This reduction is a reflection of the significant amount of damping supplied by the control. Fig. 6(b) gives the results over a smaller frequency and magnitude window indicating that the controlled magnitudes turn out to be relatively small oscillations above the magnitude of the desired transfer function. Thus a practical control transfer function $G_{FF}(s)$ may be implemented.

Another basic control scheme is shown in Fig. 7. It is a model-based output feedback control scheme requiring a more practical measurement of the system output instead of measuring the disturbance. The Euler–Maclaurin model is used to determine the controller transfer function required to provide a desired output over a broad range of frequencies. The closed loop system transfer function is given by

$$\frac{Y(s)}{F_d(s)} = \frac{G_S(s)}{1 + G_S(s)G_{FB}(s)}, \tag{29}$$

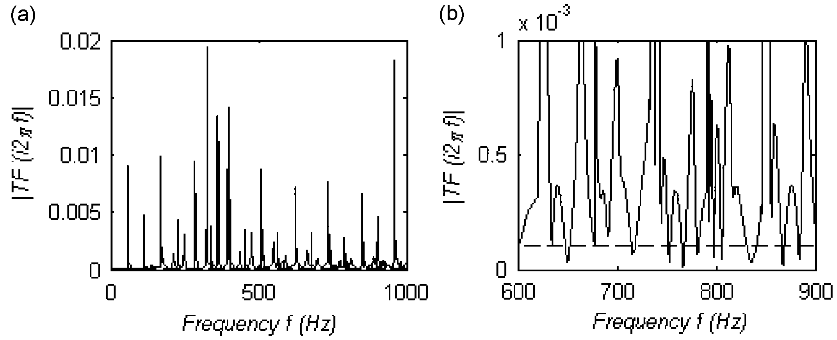

Fig. 6. Magnitude of the closed loop frequency response function for feedforward control (—) and the constant desired value of $\mathrm{TF}_{\mathrm{des}} = 0.0001$ (- - - -): (a) full frequency range and (b) smaller frequency window.

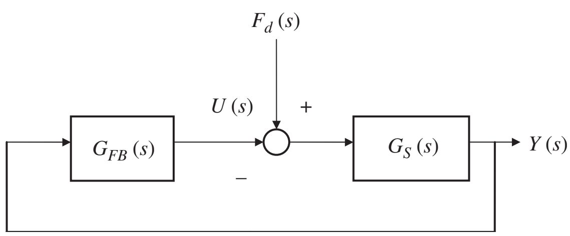

Fig. 7. Block diagram schematic for feedback control scheme.

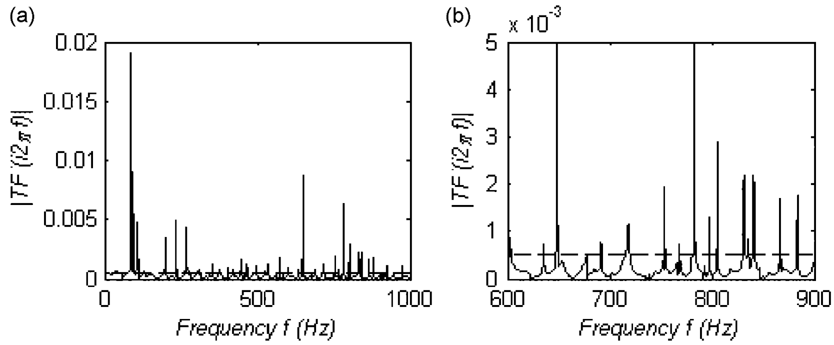

Fig. 8. Magnitude of the closed loop frequency response function for feedback control (——) and the constant desired value of $TF_{des} = 0.0005$ (- - - -): (a) full frequency range and (b) smaller frequency window.

where $G_{FB}(s)$ is the feedback controller transfer function. For the controller design, consider a desired closed loop transfer function $TF_{des}(s)$ and use the simple finite Euler–Maclaurin transfer function $G_{EM}(s)$ in Eq. (29) in place of $G_S(\omega)$ to solve for $G_{FB}(s)$. The finite controller transfer function is given by

$$G_{FB}(s) = \frac{G_{EM}(s) - TF_d(s)}{G_{EM}(s)TF_d(s)}. \tag{30}$$

The actual closed loop system transfer function is given by

$$TF_{actual}(s) = \frac{Y(s)}{F_d(s)} = \frac{G_S(s)}{1 + G_S(s)\left[\dfrac{G_{EM}(s) - TF_d(s)}{G_{EM}(s)TF_d(s)}\right]}. \tag{31}$$

As in the feedforward scheme, if one uses the actual plant transfer function as the control model plant transfer function (i.e. $G_{EM}(s) = G_S(s)$), the actual system output transfer function would equal the desired transfer function (i.e. $TF_{actual}(s) = TF_{des}(s)$). Again one has the advantage of using a finite Euler–Maclaurin sum result (with only one term for higher-order modes) for the plant transfer function in the model-based controller design to give the finite feedback controller transfer function of Eq. (30). Fig. 8(a) gives a comparison of the magnitude of the system frequency response function with control based on the finite Euler–Maclaurin sum and the desired transfer function. The desired transfer function is chosen as a flat spectrum of magnitude $|TF_{des}(s)| = 0.0005$ to drive the floor of the controlled magnitude to be in its neighborhood. The vertical scale of Fig. 8(a) is the same as the series transfer function in Fig. 4(a) to allow a direct comparison of controlled and uncontrolled systems. This comparison indicates a significant reduction of magnitudes obtained with the finite Euler–Maclaurin-based feedback controller. This reduction is a reflection of the significant amount of damping supplied by the control. Fig. 8(b) gives the results over a smaller frequency and magnitude window indicating that the controlled magnitudes turn out to be relatively small oscillations near the magnitude of the desired transfer function. Comparison of Figs. 8 and 6 indicates that the feedback control scheme provides more reduction of resonance peaks than the feedforward control scheme, even with a higher target for the desired transfer function. Also, a practical control transfer function $G_{FB}(s)$ may be implemented for the feedback scheme.

5. Remarks

The work presented here indicates that the Euler–Maclaurin sum formula can be successfully applied to provide an approximate closed-form solution for the infinite series of modal terms representing the Green's function in a 2D rectangular acoustic space. A simple form of the Euler–Maclaurin solution with only a few terms gives reasonably accurate results compared to the series solution for both time and frequency responses and for application to a transient exponential input. A comparison of oscillations and ripples relative to a plane wave component indicates that the tradeoff of using the simple less accurate Euler–Maclaurin approximate solution versus the more complex series solution may be beneficial for calculation and use in the design of controllers. Both feedforward and feedback control schemes are used to design finite controller transfer functions with the Euler–Maclaurin model which result in system transfer functions with significantly reduced resonance peaks in frequency space. The results indicate that the simple Euler–Maclaurin Green's function may be more efficient than the infinite series form for active control and optimization applications where the acoustic solution is only part of the entire mathematical problem. The approach may also be expected to provide more insight into system behavior and be applicable to other physical phenomena involving an infinite series of modal terms.

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